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Equivariant K-theory of GKM bundles

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Abstract Given a fiber bundle of GKM spaces, $\pi : M \to B$, we analyze the structure of the equivariant K-ring of M as a module over the equivariant K-ring of B by translating the fiber bundle, π, into a fiber bundle of GKM graphs and constructing, by combinatorial techniques, a basis of this module consisting of K-classes which are invariant under the natural holonomy action on the K-ring of M of the fundamental group of the GKM graph of B. We also discuss the implications of this result for fiber bundles $\pi : M \to B$ where M and B are generalized partial flag varieties and show how our GKM description of the equivariant K-ring of a homogeneous GKM space is related to the Kostant–Kumar description of this ring.

Keywords Equivariant K-theory · Equivariant fiber bundles · GKM manifolds · Flag manifolds

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1 Introduction

Let T be an n-torus and M a compact T manifold. The action of T on M is said to be GKM if M^T is finite and if, in addition, for every codimension one subtorus, T', of T the connected

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components of $M^{T'}$ are of dimension at most 2. An implication of this assumption is that these fixed-point components are either isolated points or diffeomorphic copies of S^2 with its standard S^1 action, and a convenient way of encoding this fixed-point data is by means of the *GKM graph*, Γ, of M. By definition, the S^2's above are the *edges* of this graph, the points on the fixed-point set, M^T, are the *vertices* of this graph and two vertices are joined by an edge if they are the fixed points for the S^1 action on the S^2 representing that edge. Moreover, to keep track of which T''s correspond to which edges, one defines a labeling function α on the set of oriented edges of Γ with values in the weight lattice of T. This function (the "axial function" of Γ) assigns to each oriented edge the weight of the isotropy action of T on the tangent space to the north pole of the S^2 corresponding to this edge. (The orientation of this S^2 enables one to distinguish the north pole from the south pole.)

The concept of GKM space is due to Goresky–Kottwitz–MacPherson, who showed that the equivariant cohomology ring, $H_T(M)$, can be computed from (Γ, α) (see [1]). Then Allen Knutson and Ioanid Rosu (see [6]) proved the much harder result that this is also true for the equivariant K-theory ring, $K_T(M)$. (We will give a graph theoretic description of this ring in Sect. 2 below.)

Suppose now that M and B are GKM manifolds and $\pi : M \to B$ a T-equivariant fiber bundle. Then the ring $H_T(M)$ becomes a module over the ring $H_T(B)$ and in [2] we analyzed this module structure from a combinatorial perspective by showing that the fiber bundle π of manifolds gives rise to a fiber bundle, $\pi : \Gamma_M \to \Gamma_B$ of GKM graphs, and showing that the salient module structure is encoded in this graph fiber bundle. In this article, we will prove analogous results in K-theory. More explicitly in the first part of this article (from Sects. 2 to 4) we will review the basic facts about GKM graphs, the notion of "fiber bundle" for these graphs and the definition of the K-theory ring of a graph-axial function pair, and we will also discuss an important class of examples: the graphs associated with GKM spaces of the form $M = G/P$, where G is a complex reductive Lie group and P a parabolic subgroup of G. The main results of this paper are discussed in Sects. 5 and 6. In Sect. 5 we prove that for a fiber bundle of GKM graphs $(\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ a set of elements $c_1 \ldots, c_k$ of $K_\alpha(\Gamma)$ is a free set of generators of $K_\alpha(\Gamma)$ as a module over $K_{\alpha_B}(\Gamma_B)$, providing their restrictions to $K_{\alpha_p}(\Gamma_p)$ are a basis for $K_{\alpha_p}(\Gamma_p)$, where Γ_p is the graph theoretical fiber of (Γ, α) over a vertex p of Γ_B. Then in Sect. 6 we describe an important class of such generators. One property of a GKM fiber bundle is a holonomy action of the fundamental group of Γ_B on the fiber and we show how a collection of holonomy invariant generating classes $c'_1, \ldots, c'_k$ of $K_{\alpha_p}(\Gamma_p)$ extend canonically to a free set of generators $c_1, \ldots, c_k$ of $K_\alpha(\Gamma)$. In Sects. 7 and 8 we describe how these results apply to concrete examples: special cases of the G/P examples mentioned above. Finally in Sect. 9 we relate our GKM description of the equivariant K-ring, $K_T(G/P)$, to a concise and elegant alternative description of this ring by Bertram Kostant and Shrawan Kumar in [7], and analyze from their perspective the fiber bundle $G/B \to G/P$.

To conclude these introductory remarks we would like to thank Tudor Ratiu for his support and encouragement and Tara Holm for helpful suggestions about the relations between GKM and Kostant–Kumar.

2 K-theory of integral GKM graphs

Let $\Gamma = (V, E)$ be a d-valent graph, where V is the set of vertices, and E the set of oriented edges; for every edge $e \in E$ from p to q, we denote by $\overline{e}$ the edge from q to p. Let $i : E \to V$ (respectively, $t : E \to V$) be the map which assigns to each oriented edge e its initial (respectively, terminal) vertex (so $i(e) = t(\overline{e})$ and $t(e) = i(\overline{e})$); for every $p \in V$ let E_p be the set of edges whose initial vertex is p.

Let T be an n-dimensional torus; we define a "T-action on Γ" by the following recipe(see [5]).

Definition 2.1 Let $e = (p, q)$ be an oriented edge in E. Then a *connection along* e is a bijection $\nabla_e\colon E_p \to E_q$ such that $\nabla_e(e) = \overline{e}$. A connection on the graph Γ is a family $\nabla = \{\nabla_e\}_{e\in E}$ satisfying $\nabla_{\overline{e}} = \nabla_e^{-1}$ for every $e \in \Gamma$.

Let $\mathfrak{t}^*$ be the dual of the Lie algebra of T and $\mathbb{Z}_T^*$ its weight lattice.

Definition 2.2 Let ∇ be a connection on Γ. A ∇-*compatible integral axial function* on Γ is a map $\alpha\colon E \to \mathbb{Z}_T^*$ satisfying the following conditions:

(1) $\alpha(\overline{e}) = -\alpha(e)$;
(2) for every $p \in V$ the vectors $\{\alpha(e) \mid e \in E_p\}$ are pairwise linearly independent;
(3) for every edge $e = (p, q)$ and every $e' \in E_p$ we have

$$\alpha(\nabla_e(e')) - \alpha(e') = m(e, e')\alpha(e),$$

where $m(e, e')$ is an integer which depends on e and e'.

An *integral axial function* on Γ is a map $\alpha\colon E \to \mathbb{Z}_T^*$ which is ∇-compatible, for some connection ∇ on Γ.

Definition 2.3 An *integral GKM graph* is a pair (Γ, α) consisting of a regular graph Γ and an integral axial function $\alpha\colon E \to \mathbb{Z}_T^*$

Remark 2.4 The graphs we described in the introduction are examples of such graphs. In particular condition 2 in Definition 2.2 is a consequence of the fact that, for every codimension one subgroup of T, its fixed-point components are of dimension at most two, and condition 3 a consequence of the fact that this subgroup acts trivially on the tangent bundles of these component.

Observe that an integral GKM graph is a particular case of an abstract GKM graph, as defined in [2]; here we require α to take values in $\mathbb{Z}_T^*$ rather than in $\mathfrak{t}^*$, and in Definition 2.2 (3) we require $\alpha(\nabla_e(e')) - \alpha(e')$ to be an integer multiple of $\alpha(e)$, for every $e = (p, q) \in E$ and $e' \in E_p$. The necessity of these integrality properties will be clear from the definition of *T-action on* Γ. Let $R(T)$ be the representation ring of T; notice that $R(T)$ can be identified with the character ring of T, i.e., with the ring of finite sums

$$\sum_k m_k e^{2\pi\sqrt{-1}\alpha_k}, \tag{2.1}$$

where the m_k's are integers and $\alpha_k \in \mathbb{Z}_T^*$. So giving an axial function $\alpha\colon E \to \mathbb{Z}_T^*$ is equivalent to giving a map which assigns to each edge $e \in E$ a one dimensional representation ρ_e, whose character $\chi_e\colon T \to S^1$ is given by

$$\chi_e\left(e^{2\pi\sqrt{-1}\xi}\right) = e^{2\pi\sqrt{-1}\alpha(e)(\xi)}.$$

For every $e \in E$, let $T_e = \ker(\chi_e)$, and consider the restriction map

$$r_e\colon R(T) \to R(T_e).$$

Then for every vertex $p \in V$, we also obtain a d-dimensional representation

$$\nu_p \simeq \bigoplus_{e \in E_p} \rho_e$$

which, by Definition 2.2 (3) satisfies

$$r_e(\nu_{i(e)}) \simeq r_e(\nu_{t(e)}). \tag{2.2}$$

So an integral axial function $\alpha\colon E \to \mathbb{Z}_T^*$ defines a one dimensional representation ρ_e for every edge $e \in E$ and for every $p \in V$ a d-dimensional representation ν_p satisfying (2.2); this is what we refer to as a *T-action on* Γ.

Remark 2.5 Henceforth in this article all GKM graphs will be, unless otherwise specified, *integral* GKM graphs.

We will now define the K-ring $K_\alpha(\Gamma)$ of (Γ, α). As we remarked in the introduction, Knutson and Rosu have proved that if (Γ, α) is the GKM graph associated to a GKM manifold M, then

$$K_\alpha(\Gamma) \simeq K_T(M),$$

where $K_T(M)$ is the equivariant K-theory ring of M (cf. [6]).

Let $\mathrm{Maps}(V, R(T))$ be the ring of maps which assign to each vertex $p \in V$ a representation of T. Following the argument in [6], we define a subring of $\mathrm{Maps}(V, R(T))$, called the ring of *K-classes of* (Γ, α).

Definition 2.6 Let f be an element of $\mathrm{Maps}(V, R(T))$. Then f is a K-class of (Γ, α) if for every edge $e = (p, q) \in E$

$$r_e(f(p)) = r_e(f(q)). \tag{2.3}$$

Observe that using the identification of $R(T)$ with the ring of finite sums (2.1), condition (2.3) is equivalent to saying that for every $e = (p, q) \in E$

$$f(p) - f(q) = \beta\left(1 - e^{2\pi\sqrt{-1}\alpha(e)}\right), \tag{2.4}$$

for some β in $R(T)$.

If f and g are two K-classes, then also $f + g$ and fg are; so the set of K-classes is a subring of $\mathrm{Maps}(V, R(T))$.

Definition 2.7 The *K-ring* of (Γ, α), denoted by $K_\alpha(\Gamma)$, is the subring of $\mathrm{Maps}(V, R(T))$ consisting of all the K-classes.

3 GKM fiber bundles

Let (Γ_1, α_1) and (Γ_2, α_2) be GKM graphs, where $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$, $\alpha_1\colon E_1 \to \mathbb{Z}_{T_1}^* \subset \mathfrak{t}_1^*$ and $\alpha_2\colon E_2 \to \mathbb{Z}_{T_2}^* \subset \mathfrak{t}_2^*$.

Definition 3.1 An *isomorphism of GKM graphs* (Γ_1, α_1) and (Γ_2, α_2) is a pair (Φ, Ψ), where $\Phi\colon \Gamma_1 \to \Gamma_2$ is an isomorphism of graphs, and $\Psi\colon \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ is an isomorphism of linear spaces such that $\Psi(\mathbb{Z}_{T_1}^*) = \mathbb{Z}_{T_2}^*$, and for every edge (p, q) of Γ_1 we have $\alpha_2(\Phi(p), \Phi(q)) = \Psi(\alpha_1(p, q))$.

By definition, if (Φ, Ψ) is an isomorphism of GKM graphs, the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\alpha_1"'] & E_2 \arrow[d, "\alpha_2"] \\ \mathbb{Z}^*_{T_1} \arrow[r, "\Psi|_{\mathbb{Z}^*_{T_1}}"] & \mathbb{Z}^*_{T_2} \end{tikzcd} \tag{3.1}$$

commutes. We can extend the map Ψ to be a ring homomorphism from $R(T_1)$ to $R(T_2)$ using the identification (2.1) and defining $\Psi\left(e^{2\pi\sqrt{-1}\alpha}\right) = e^{2\pi\sqrt{-1}\Psi(\alpha)}$, for every $\alpha \in \mathbb{Z}^*_{T_1}$.

Given $f \in \mathrm{Maps}(V_2, R(T_2))$, let $\Upsilon^*(f) \in \mathrm{Maps}(V_1, R(T_1))$ be the map defined by $\Upsilon^*(f)(p) = \Psi^{-1}(f(\Phi(p)))$. From the commutativity of the diagram (3.1) it's easy to see that if $f \in K_{\alpha_2}(\Gamma_2)$ then $\Upsilon^*(f) \in K_{\alpha_1}(\Gamma_1)$, and Υ^* defines an isomorphism between the two K-rings.

We are now ready to define the main combinatorial objects of this paper, GKM fiber bundles. Let (Γ, α) and (Γ_B, α_B) be two GKM graphs, with α and α_B having images in the same weight lattice $\mathbb{Z}^*_T$. Let $\pi : \Gamma = (V, E) \to \Gamma_B = (V_B, E_B)$ be a surjective morphism of graphs. By that we mean that π maps the vertices of Γ onto the vertices of Γ_B, such that, for every edge $e = (p, q)$ of Γ, either $\pi(p) = \pi(q)$ (in which case e is called *vertical*), or $(\pi(p), \pi(q))$ is an edge of Γ_B (in which case e is called *horizontal*). Such a morphism of graphs induces a map $(d\pi)_p : H_p \to E_{\pi(p)}$ from the set of horizontal edges at $p \in V$ to the set of all edges starting at $\pi(p) \in V_B$. The first condition we impose for π to be a GKM fiber bundle is the following:

1: For all vertices $p \in V$, $(d\pi)_p : H_p \to E_{\pi(p)}$ is a bijection compatible with the axial functions:

$$\alpha_B((d\pi)_p(e)) = \alpha(e),$$

for all $e = (p, q) \in H_p$.

The second condition has to do with the connections on Γ and Γ_B.

2: The connection along edges of Γ moves horizontal edges to horizontal edges, and vertical edges to vertical edges. Moreover, the restriction of the connection of Γ to horizontal edges is compatible with the connection on Γ_B.

For every vertex $p \in \Gamma_B$, let $V_p = \pi^{-1}(p) \subset V$ and Γ_p the induced subgraph of Γ with vertex set V_p. If the map π satisfies condition **1**, then, for every edge $e = (p, q)$ of Γ_B, it induces a bijection $\Phi_{p,q} : V_p \to V_q$ by $\Phi_{p,q}(p') = q'$ if and only if $(p, q) = (d\pi)_p(p', q')$.

3: For every edge $(p, q) \in \Gamma_B$, $\Phi_{p,q} : \Gamma_p \to \Gamma_q$ is an isomorphism of graphs compatible with the connection ∇ on Γ in the following sense: for every lift $e' = (p_1, q_1)$ of $e = (p, q)$ at p_1 and every edge $e'' = (p_1, p_2)$ of Γ_p the connection along the horizontal edge (p_1, q_1) moves the vertical edge (p_1, p_2) to the vertical edge (q_1, q_2), where $q_i = \Phi_{p,q}(p_i)$, $i = 1, 2$.

We can endow Γ_p with a GKM structure, which is just the restriction of the GKM structure of (Γ, α) to Γ_p. The axial function on Γ_p is the restriction of $\alpha : E \to \mathbb{Z}^*_T$ to the edges of Γ_p; we refer to it as α_p, and it takes values in $\mathfrak{v}^*_p$, the subspace of $\mathfrak{t}^*$ generated by values of axial functions $\alpha(e)$, for edges e of Γ_p. The next condition we impose on π is the following:

4: For every edge (p, q) of Γ_B, there exists an isomorphism of GKM graphs

$$\Upsilon_{p,q} = (\Phi_{p,q}, \Psi_{p,q})\colon (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q).$$

By property (3) of Definition 2.2 and the fact that $\alpha_B(e) = \alpha(e')$ we have

$$\alpha(\Phi_{p,q}(p_1), \Phi_{p,q}(p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e), \tag{3.2}$$

where $m'(p_1, p_2)$ is an integer; so by the commutativity of (3.1) we have

$$\Psi_{p,q}(\alpha(p_1, p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e). \tag{3.3}$$

Condition (3.3) defines a map $m'\colon E_p \to \mathbb{Z}$, where E_p is the set of edges of Γ_p. The next condition is a strengthening of this.

5: There exists a function $m\colon \mathfrak{v}_p^* \cap \mathbb{Z}_T^* \to \mathbb{Z}$ such that

$$\Psi_{p,q}(x) = x + m(x)\alpha_B(p, q). \tag{3.4}$$

Observe that if Γ_B is connected, then all the fibers of π are isomorphic as GKM graphs. More precisely, for any two vertices $p, q \in V_B$ let γ be a path in Γ_B from p to q, i.e., $\gamma\colon p = p_0 \to p_1 \to \cdots \to p_m = q$. Then the map

$$\Upsilon_\gamma = \Upsilon_{p_{m-1},p_m} \circ \cdots \circ \Upsilon_{p_0,p_1}\colon (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q).$$

defines an isomorphism between the GKM graphs (Γ_p, α_p) and (Γ_q, α_q). As observed before, this isomorphism restricts to an isomorphism between the two K-rings, $\Upsilon_\gamma^*\colon K_{\alpha_q}(\Gamma_q) \to K_{\alpha_p}(\Gamma_p)$, which is not an isomorphism of $R(T)$-modules, unless the linear isomorphism

$$\Psi_\gamma = \Psi_{p_0,p_1} \circ \cdots \circ \Psi_{p_{m-1},p_m}\colon \mathfrak{v}_q^* \to \mathfrak{v}_p^*$$

is the identity.

Let $\Omega(p)$ be the set of all loops in Γ_B that start and end at p, i.e., the set of paths $\gamma\colon p_0 \to p_1 \to \cdots \to p_{m-1} \to p_m$ such that $p_0 = p_m = p$. Then every such γ determines a GKM isomorphism Υ_γ of the fiber (Γ_p, α_p). The *holonomy group* of the fiber (Γ_p, α_p) is the subgroup of the GKM isomorphisms of the fiber, $\mathrm{Aut}(\Gamma_p, \alpha_p)$, given by

$$\mathrm{Hol}_\pi(\Gamma_p) = \{\Upsilon_\gamma \mid \gamma \in \Omega(p)\} \leqslant \mathrm{Aut}(\Gamma_p, \alpha_p).$$

4 Flag manifolds as GKM fiber bundles

In this section, we will discuss some important examples of GKM fiber bundles coming from generalized partial flag varieties.

Let G be a complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\mathfrak{t} \subset \mathfrak{h}$ a compact real form; let T be the compact torus whose Lie algebra is $\mathfrak{t}$. Let $\Delta \subset \mathbb{Z}_T^* \subset \mathfrak{t}^*$ be the set of roots and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

the Cartan decomposition of $\mathfrak{g}$. Let $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta$ be a choice of simple roots and Δ^+ the corresponding positive roots. The set of positive roots determines a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ given by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

Let $\Sigma \subset \Delta_0$ be a subset of simple roots, and $B \leqslant P(\Sigma) \leqslant G$, where B is the Borel subgroup whose Lie algebra is $\mathfrak{b}$ and $P(\Sigma)$ the parabolic subgroup of G corresponding to Σ. Then $M = G/P(\Sigma)$ is a (generalized) partial flag manifold. In particular, if $\Sigma = \emptyset$, then $P(\emptyset) = B$, and we refer to $M = G/B$ as the (generalized) complete flag manifold.

The torus T with Lie algebra $\mathfrak{t}$ acts on $M = G/P(\Sigma)$ by left multiplication on G; this action determines a GKM structure on M with GKM graph (Γ, α). In fact, let W be the Weyl group of $\mathfrak{g}$ and $W(\Sigma)$ the subgroup of W generated by reflections across the simple roots in Σ, and $\langle\Sigma\rangle$ the positive roots which can be written as a linear combination of the roots in Σ. Then the vertices of Γ, corresponding to the T-fixed points, are in bijection with the right cosets

$$W/W(\Sigma) = \{vW(\Sigma) \mid v \in W\} = \{[v] \mid v \in W\},$$

where $[v] = vW(\Sigma)$ is the right $W(\Sigma)$-coset containing v. Two vertices $[v]$ and $[w]$ are joined by an edge if and only if there exists $\beta \in \Delta^+ \setminus \langle\Sigma\rangle$ such that $[v] = [ws_\beta]$; moreover the axial function α on the edge $e = ([v], [vs_\beta])$ is given by $\alpha([v], [vs_\beta]) = v\beta$; and it's easy to see that this label is well defined. Moreover, given an edge $e' = ([v], [vs_{\beta'}])$ starting at $[v]$, a natural choice of a connection along e is given by $\nabla_e e' = ([vs_\beta], [vs_\beta s_{\beta'}])$. Then

$$\alpha(\nabla_e e') - \alpha(e') = vs_\beta\beta' - v\beta' = s_{v\beta}v\beta' - v\beta' = m(e, e')v\beta,$$

where $m(e, e')$ is a Cartan integer; so this connection satisfies property (3) of Definition 2.2.

Consider the natural T-equivariant projection $\pi : G/B \to G/P(\Sigma)$; this map induces a projection map on the corresponding GKM graphs $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ given by $\pi(w) = [w]$ for every $w \in W$, where $[w] = wW(\Sigma)$. As we proved in [2, section 4.3], π is a GKM fiber bundle. it is in fact a balanced bundle,(see [3]).Moreover, let Γ_0 be the GKM graph of the fiber containing the identity element of W; then

$$\mathrm{Hol}_\pi(\Gamma_0) \simeq W(\Sigma).$$

The key ingredient in the proof of this is the identification of the weights of the T action on the tangent space to the identity coset p_0 of G/P with the complement of $\langle\Sigma\rangle$ in Δ^+. Namely for any such root α let $w \in W$ be the Weyl group element $w : \mathfrak{t} \to \mathfrak{t}$ associated with the reflection in the hyperplane $\alpha = 0$. Then for the edge $e = (p_0, wp_0)$ of the GKM graph of G/P the GKM isomorphism of Γ_{p_0} onto Γ_p is the isomorphism $\Phi_{p_0,p}$ associated with the left action of w on G/B and the T automorphism (3.4) is just the action of w on $\mathfrak{t}$ given by compositions of reflections in the hyperplane $\alpha_B(p, q) = 0$ in $\mathfrak{t}$, for some horizontal edge (p, q). (These results are also a byproduct of the identification of the GKM model of $K_T(G/P)$ with the Kostant–Kumar model which we will describe in Sect. 9).

5 K-theory of GKM fiber bundles

Given a GKM fiber bundle $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$, we will describe the K-ring of (Γ, α) in terms of the K-ring of (Γ_B, α_B). In the proof of the main theorem we will need the following technical lemma.

Lemma 5.1 *Let α and β be linearly independent weights in $\mathfrak{t}^*$, and P an element of $R(T)$. If $1 - e^{2\pi\sqrt{-1}\alpha}$ divides $\left(1 - e^{2\pi\sqrt{-1}\beta}\right) P$ then $1 - e^{2\pi\sqrt{-1}\alpha}$ divides P.*

Proof Let $\alpha = m\alpha_1$, for some $m \in \mathbb{Z}$, where α_1 is a primitive element of the weight lattice in $\mathfrak{t}^*$. We can complete α_1 to a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ of the lattice. Let $x_j = e^{2\pi\sqrt{-1}\alpha_j}$ for

all $j = 1, \dots, d$.
Then by hypothesis $(1 - x_1^m)$ divides $(1 - x_1^n Q(x_2, \dots, x_d))P(x_1, \dots, x_d)$ for some non-constant polynomial $Q(x_2, \dots, x_d)$. Consider an element $\xi \in \mathfrak{t} \otimes \mathbb{C}$ such that $x_1(\xi)^m = 1$; then $(1 - x_1^n Q(x_2, \dots, x_d))P(x_1, \dots, x_d)(\xi) = 0$. Since in general $(1 - x_1^n Q(x_2, \dots, x_d))(\xi) \neq 0$, this implies that $(1 - x_1^m)$ divides $P(x_1, \dots, x_d)$. $\square$

For every K-class $f \colon V_B \to R(T)$, define the pull-back $\pi^*(f) \colon V \to R(T)$ by $\pi^*(f)(q) = f(\pi(q))$. It's easy to check that $\pi^*(f)$ is a K-class on (Γ, α). So $K_\alpha(\Gamma)$ contains $K_{\alpha_B}(\Gamma_B)$ as a subring, and the map $\pi^* \colon K_{\alpha_B}(\Gamma_B) \to K_\alpha(\Gamma)$ gives $K_\alpha(\Gamma)$ the structure of a $K_{\alpha_B}(\Gamma_B)$-module.

Definition 5.2 A K-class $h \in K_\alpha(\Gamma)$ is called *basic* if $h \in \pi^*(K_{\alpha_B}(\Gamma_B))$.

We denote the subring of basic K-classes by $(K_\alpha(\Gamma))_{bas}$; clearly we have

$$(K_\alpha(\Gamma))_{bas} \simeq K_{\alpha_B}(\Gamma_B).$$

Theorem 5.3 *Let $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ be a GKM fiber bundle, and let $c_1, \dots, c_m$ be K-classes on Γ such that for every $p \in V_B$ the restriction of these classes to the fiber $\Gamma_p = \pi^{-1}(p)$ form a basis for the K-ring of the fiber. Then, as $K_{\alpha_B}(\Gamma_B)$-modules, $K_\alpha(\Gamma)$ is isomorphic to the free $K_{\alpha_B}(\Gamma_B)$-module on $c_1, \dots, c_m$.*

Proof First of all, observe that any linear combination of $c_1, \dots, c_m$ with coefficients in $(K_\alpha(\Gamma))_{bas} \simeq K_{\alpha_B}(\Gamma_B)$ is an element of $K_\alpha(\Gamma)$. Now we want to prove that the c_i's are independent over $K_{\alpha_B}(\Gamma_B)$. In order to prove so, let $\sum_{k=1}^m \beta_k c_k = 0$ for some $\beta_1, \dots, \beta_m \in (K_\alpha(\Gamma))_{bas}$. Let $\Gamma_p = \pi^{-1}(p)$ denote the fiber over $p \in B$, $\iota_p : \Gamma_p \to \Gamma$ the inclusion, and $\iota_p^* \colon K_\alpha(\Gamma) \to K_\alpha(\Gamma_p)$ the restriction to the K-theory of the fiber. Then $\sum_{k=1}^m \iota_p^*(\beta_k c_k) = 0$ for all $p \in B$. Since the β_k's are basic K-classes, $\iota_p^*(\beta_k)$ is just an element of $R(T)$ for all k. But by assumption $\{\iota_p^*(c_1), \dots, \iota_p^*(c_m)\}$ is a basis of $K_\alpha(\Gamma_p)$; so $\iota_p^*(\beta_k) = 0$ for all $k = 1, \dots, m$, for all $p \in \Gamma_B$, which implies that $\beta_k = 0$ for all k. We need to prove that the free $K_{\alpha_B}(\Gamma_B)$-module generated by $c_1, \dots, c_m$ is $K_\alpha(\Gamma)$.

Let $c \in K_\alpha(\Gamma)$. Since the classes $\iota_p^* c_1, \dots, \iota_p^* c_m$ are a basis for $K_\alpha(\Gamma_p)$, there exist $\beta_1, \dots, \beta_m \in \mathrm{Maps}(B, R(T))$ such that

$$c = \sum_{k=1}^m \beta_k c_k;$$

we need to prove that the β_k's belong to $(K_\alpha(\Gamma))_{bas}$ for all k. In order to prove this, it is sufficient to show that (2.4) is satisfied for every edge (p, q) of Γ_B. Let $e' = (p', q')$ be the lift of (p, q) at $p' \in \Gamma_p$. Then

$$\begin{aligned} c(q') - c(p') &= \sum_{k=1}^m (\beta_k(q) c_k(q') - \beta_k(p) c_k(p')) \\ &= \sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') + \sum_{k=1}^m \beta_k(q)(c_k(q') - c_k(p')). \end{aligned}$$

Since $c, c_1, \dots, c_m$ belong to $K_\alpha(\Gamma)$, by (2.4) the differences $c(q') - c(p')$, $c_k(q') - c_k(p')$ are multiples of $(1 - e^{2\pi\sqrt{-1}\alpha(e')})$, for all $k = 1, \dots, m$. Therefore, for all $p' \in \Gamma_p$,

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') = \left(1 - e^{2\pi\sqrt{-1}\alpha(e')}\right) \eta(p'),$$

where $\eta(p') \in R(T)$. We will show that $\eta\colon \Gamma_p \to R(T)$ belongs to $K_\alpha(\Gamma_p)$.

If p' and p'' are vertices in Γ_p, joined by an edge (p', p''), then

$$\sum_{k=1}^{m}(\beta_k(q)-\beta_k(p))(c_k(p'')-c_k(p')) = \left(1-e^{2\pi\sqrt{-1}\alpha(e')}\right)(\eta(p'')-\eta(p')).$$

Each c_k is a K-class on Γ, so $c_k(p'')-c_k(p')$ is a multiple of $\left(1-e^{2\pi\sqrt{-1}\alpha(p',p'')}\right)$, for all $k=1,\ldots,m$. Then $\left(1-e^{2\pi\sqrt{-1}\alpha(p',p'')}\right)$ divides $\left(1-e^{2\pi\sqrt{-1}\alpha(e')}\right)(\eta(p'')-\eta(p'))$. But $\alpha(e')$ and $\alpha(p', p'')$ are linearly independent vectors. Therefore, by Lemma 5.1, $\left(1-e^{2\pi\sqrt{-1}\alpha(p',p'')}\right)$ divides $\eta(p'')-\eta(p')$, and so η is a K-class on Γ_p.

Since the classes $\iota_p^*c_1,\ldots,\iota_p^*c_m$ are a basis for $K_\alpha(\Gamma_p)$ there exist $Q_1,\ldots,Q_m \in R(T)$ such that

$$\eta = \sum_{k=1}^{m} Q_k\iota_p^*c_k.$$

Then

$$\sum_{k=1}^{m}\left(\beta_k(q)-\beta_k(p)-Q_k\left(1-e^{2\pi\sqrt{-1}\alpha(e')}\right)\right)\iota_p^*c_k = 0.$$

But $\iota_p^*c_1,\ldots,\iota_p^*c_m$ are linearly independent over $R(T)$, so

$$\beta_k(q)-\beta_k(p) = Q_k\left(1-e^{2\pi\sqrt{-1}\alpha(e')}\right).$$

Since $\alpha(e')=\alpha(p',q')=\alpha_B(p,q)$, this implies that $\beta_k \in K_{\alpha_B}(\Gamma_B)$. Therefore every K-class on Γ can be written as a linear combination of classes $c_1,\ldots,c_m$, with coefficients in $K_{\alpha_B}(\Gamma_B)$. □

6 Invariant classes

Let $\pi:(\Gamma,\alpha)\to(\Gamma_B,\alpha_B)$ be a GKM fiber bundle, and let (Γ_p,α_p) be one of its fibers. We say that $f \in K_{\alpha_p}(\Gamma_p)$ is an *invariant class* if $\Upsilon_\gamma^*(f)=f$ for every $\Upsilon_\gamma \in \mathrm{Hol}_\pi(\Gamma_p)$. We denote by $(K_{\alpha_p}(\Gamma_p))^{\mathrm{Hol}}$ the subring of $K_{\alpha_p}(\Gamma_p)$ given by invariant classes.

Given any such class $f \in (K_{\alpha_p}(\Gamma_p))^{\mathrm{Hol}}$, we can extend it to be an element of Maps: $V \to R(T)$ by the following recipe: let q be a vertex of Γ_B, and γ a path in Γ_B from q to p. Let $\Upsilon_\gamma\colon(\Gamma_q,\alpha_q)\to(\Gamma_p,\alpha_p)$ be the isomorphism of GKM graphs associated to γ; then, as we observed before, $\Upsilon_\gamma^*(f)$ defines an element of $K_{\alpha_q}(\Gamma_q)$. Notice that the invariance of f implies that $\Upsilon_\gamma^*(f)$ only depends on the end-points of γ; so we denote $\Upsilon_\gamma^*(f)$ by f_q.

Proposition 6.1 *Let $c \in \mathrm{Maps}(V, R(T))$ be the map defined by $c(q')=f_{\pi(q')}(q')$ for any $q' \in V$. Then $c \in K_\alpha(\Gamma)$.*

Proof Since c is a class on each fiber, it is sufficient to check the compatibility Condition (2.4) on horizontal edges; let $e=(p,q)$ be an edge of Γ_B, and let $e'=(p',q')$ be its lift at $p' \in V$.

If $c(p') = f_p(p') = \sum_k n_k e^{2\pi\sqrt{-1}\alpha_k}$, where $\alpha_k \in \mathbb{Z}_T^* \cap \mathfrak{v}_p^*$ and $n_k \in \mathbb{Z}$ for all k, then

$$c(p') - c(q') = f_p(p') - f_q(q') = f_p(p') - \Psi_e(f_p(p'))$$
$$= \sum_k n_k \left(e^{2\pi\sqrt{-1}\alpha_k} - e^{2\pi\sqrt{-1}\Psi_e(\alpha_k)}\right).$$

By definition of GKM fiber bundle, $\Psi_e(\alpha_k) = \alpha_k + c(\alpha_k)\alpha_B(p,q)$ for every k, where $c(\alpha_k)$ is an integer and $\alpha_B(p,q) = \alpha(p',q')$. So $e^{2\pi\sqrt{-1}\alpha_k} - e^{2\pi\sqrt{-1}\Psi_e(\alpha_k)} = e^{2\pi\sqrt{-1}\alpha_k}\left(1 - e^{2\pi\sqrt{-1}c(\alpha_k)\alpha_B(p,q)}\right)$, and it's easy to see that $\left(1 - e^{2\pi\sqrt{-1}c(\alpha_k)\alpha_B(p,q)}\right) = \beta_k\left(1 - e^{2\pi\sqrt{-1}\alpha_B(p,q)}\right)$ for some $\beta_k \in R(T)$, for every k. □

7 Classes on projective spaces

Let $T = (S^1)^n$ be the compact torus of dimension n, with Lie algebra $\mathfrak{t} = \mathbb{R}^n$, and let $\{y_1, \ldots, y_n\}$ be the basis of $\mathfrak{t}^* \simeq \mathbb{R}^n$ dual to the canonical basis of $\mathbb{R}^n$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. The torus T acts componentwise on $\mathbb{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n). \tag{7.1}$$

This action induces a GKM action of T on $\mathbb{C}P^{n-1}$, and the GKM graph is $\Gamma = \mathcal{K}_n$, the complete graph on n vertices labelled by $[n] = \{1, \ldots, n\}$. The axial function α on the edge (i, j) is $y_i - y_j$, for every $i \neq j$. Let $\mathbb{S} = \mathbb{Z}[y_1, \ldots, y_n]$, $(\mathbb{S})$ the field of fractions of $\mathbb{S}$, $\mathcal{M} = \mathrm{Maps}([n], \mathbb{S})$, and

$$H_\alpha(\Gamma) = \{f \in \mathcal{M} \mid f(j) - f(k) \in (y_j - y_k)\mathbb{S},\ \text{for all } j \neq k\}.$$

Then $H_\alpha(\Gamma)$ is an $\mathbb{S}$-subalgebra of $\mathcal{M}$. Let $\int_\Gamma : \mathcal{M} \to (\mathbb{S})$ be the map

$$\int_\Gamma f = \sum_{k=1}^n \frac{f(k)}{\prod_{j\neq k}(y_k - y_j)}.$$

Proposition 7.1 *Let $f \in \mathcal{M}$. Then $f \in H_\alpha(\Gamma)$ if and only if $\int_\Gamma f \in \mathbb{S}$.*

Proof We have

$$\int_\Gamma f = \sum_{k=1}^n \frac{f(k)}{\prod_{j\neq k}(y_k - y_j)} = \frac{P}{\prod_{j<k}(y_j - y_k)},$$

where $P \in \mathbb{S}$. The factors in the denominator are distinct and relatively prime, hence $\int_\Gamma f \in \mathbb{S}$ if and only if all factors in the denominator divide P.

The factor $y_j - y_k$ comes from

$$\frac{f(k)}{\prod_{i\neq k}(y_i - y_j)} + \frac{f(j)}{\prod_{i\neq j}(y_i - y_j)} = \frac{f(j) - f(k)}{(y_j - y_k)\prod_{i\neq j,k}(y_j - y_i)} +$$
$$+ \frac{f(k)\left(\prod_{i\neq k,k}(y_k - y_i) - \prod_{i\neq j,k}(y_j - y_i)\right)}{(y_j - y_k)\prod_{i\neq j,k}(y_j - y_i)(y_k - y_i)}.$$

But $y_j - y_k$ divides the numerator of the second fraction, hence $y_j - y_k$ divides P if and only if it divides $f(j) - f(k)$. □

The permutation group S_n acts on $\mathbb{S}$ by permuting variables, and that action induces an action on $H_\alpha(\Gamma)$ by

$$(w \cdot f)(j) = w^{-1} \cdot f(w(j)).$$

We say that a class $f \in H_\alpha(\Gamma)$ is S_n*-invariant* if $w \cdot f = f$ for every $w \in S_n$, i.e.,

$$f(w(j)) = w \cdot f(j) \quad \text{for every} \quad w \in S_n.$$

The goal of this section is to construct bases of the $\mathbb{S}$-module $H_\alpha(\Gamma)$ consisting of S_n-invariant classes, and to give explicit formulas for the coordinates of a given class in those bases.

Let $\phi\colon [n] \to \mathbb{S}$, $\phi(j) = y_j$ for all $1 \leqslant j \leqslant n$. Then ϕ is an S_n-invariant class in $H_\alpha(\Gamma)$. For $1 \leqslant k \leqslant n$, let $f_k = \phi^{k-1}$. Then $f_1, f_2, \ldots, f_n$ are $\mathbb{S}$-linearly independent invariant classes.

For $0 \leqslant j \leqslant n$, let s_j be the jth elementary symmetric polynomial in the variables $y_1, \ldots, y_n$. Then $s_0 = 1, s_1 = y_1 + \cdots + y_n, s_2 = y_1y_2 + y_1y_3 + \cdots + y_{n-1}y_n, \ldots,$ $s_n = y_1y_2 \cdots y_n$. For $1 \leqslant k \leqslant n$, let

$$g_k = f_k - s_1 f_{k-1} + s_2 f_{k-2} - \cdots + (-1)^{k-1} s_{k-1} f_1.$$

Then $g_1, g_2, \ldots, g_n$ are invariant classes and the transition matrix from the f's to the g's is triangular with ones on the diagonal, hence it is invertible over $\mathbb{S}$. Therefore the classes $g_1, \ldots, g_n$ are also $\mathbb{S}$-linearly independent.

Let $\langle ., . \rangle\colon H_\alpha(\Gamma) \times H_\alpha(\Gamma) \to \mathbb{S}$ be the pairing

$$\langle f, g \rangle = \int\limits_\Gamma fg.$$

Theorem 7.2 *The sets of classes* $\{f_1, \ldots, f_n\}$ *and* $\{g_n, \ldots, g_1\}$ *are dual to each other:*

$$\langle f_j, g_{n-k+1} \rangle = \delta_{jk},$$

for all $1 \leqslant j, k \leqslant n$.

Proof We have $\int_\Gamma \phi^k = 0$ for all $0 \leqslant k \leqslant n-2$ and $\int_\Gamma \phi^{n-1} = 1$. Moreover

$$\phi^n - s_1\phi^{n-1} + s_2\phi^{n-2} - \ldots + (-1)^n s_n \phi^0 = 0.$$

Let $1 \leqslant j, k \leqslant n$. Then

$$f_j g_{n-k+1} = \phi^{j-1} \left(\phi^{n-k} - s_1 \phi^{n-k-1} + \cdots + (-1)^{n-k} s_{n-k} \phi^0 \right).$$

If $j < k$, then

$$f_j g_{n-k+1} = \text{ a combination of powers of } \phi \text{ at most } n-2,$$

and then $\langle f_j, g_{n-k+1} \rangle = 0$.

If $j = k$, then

$$f_k g_{n-k+1} = \phi^{n-1} + \text{ a combination of powers of } \phi \text{ at most } n-2,$$

hence $\langle f_k, g_{n-k+1} \rangle = \int_\Gamma \phi^{n-1} = 1$.

If $j > k$, then

$$\begin{aligned} f_j g_{n-k+1} &= \phi^{j-k-1}\left(\phi^n - s_1\phi^{n-1} + \cdots + (-1)^{n-k}s_{n-k}\phi^k\right) \\ &= -\phi^{j-k-1}\left((-1)^{n-k-1}s_{n-k+1}\phi^{k-1} + \ldots + (-1)^n s_n\phi^0\right) \\ &= \text{ a combination of powers of } \phi \text{ at most } n-2, \end{aligned}$$

and then $\langle f_j, g_{n-k+1}\rangle = 0$. □

The following is an immediate consequence of this result.

Corollary 7.3 *The sets* $\{f_1, f_2, \ldots, f_n\}$ *and* $\{g_1, g_2, \ldots, g_n\}$ *are dual bases of the* $\mathbb{S}$*-module* $H_\alpha(\Gamma)$*, and both bases consist of invariant classes. Moreover, if* $h \in H_\alpha(\Gamma)$ *and*

$$h = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = b_1 g_1 + b_2 g_2 \cdots + b_n g_n,$$

then

$$a_k = \langle g_{n-k+1}, h\rangle \quad \textit{and} \quad b_k = \langle f_{n-k+1}, h\rangle.$$

The entire discussion above extends naturally to K-theory. Let $z_j = e^{2\pi\sqrt{-1}y_j}$ for $j = 1, \ldots, n$. If $y_1, \ldots, y_n$ denote a basis of $\mathbb{Z}_T^*$, we have that

$$R(T) = \mathbb{Z}\left[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}\right].$$

Let $\psi\colon \mathbb{S} \to R(T)$ be the injective ring morphism determined by $\psi(y_j) = z_j$, for $j = 1, \ldots, n$. Its image is $\psi(\mathbb{S}) = R_+(T) = \mathbb{Z}[z_1, \ldots, z_n]$. Let

$$K_\alpha(\Gamma) = \{g\colon [n] \to R(T) \mid f(j) - f(k) \in (z_j - z_k)R(T), \text{ for all } j \neq k\}.$$

Then

$$\Phi\colon H_\alpha(\Gamma) \to K_\alpha(\Gamma) \quad , \quad \Phi(f)(j) = \psi(f(j))$$

is an injective morphism of rings and

$$\Phi(qf) = \psi(q)\Phi(f)$$

for all $f \in H_\alpha(\Gamma)$ and $q \in \mathbb{S}$. The image of Φ is

$$\mathrm{im}\Phi = \{g \in K_\alpha(\Gamma) \mid \mathrm{im}(g) \subset R_+(T)\}.$$

Let $\nu = \Phi(\phi)$; then $\nu\colon [n] \to R(T)$, $\nu(j) = z_j$.

The symmetric group S_n acts on $R(T)$ by simultaneously permuting the variables z and z^{-1}, and that action induces an action on $K_\alpha(\Gamma)$. The K-class ν is invariant, and so are its powers.

Proposition 7.4 *The invariant classes* $\{1, \nu, \ldots, \nu^{n-1}\}$ *form a basis of* $K_\alpha(\Gamma)$ *over* $R(T)$.

Proof A Vandermonde determinant argument shows that the classes are independent. If $g \in K_\alpha(\Gamma)$, then there exists an invertible element $u \in R(T)$ such that $ug \in \mathrm{im}\Phi$. Let $ug = \Phi(f)$, with $f \in H_\alpha(\Gamma)$. If

$$f = a_0\phi^0 + \cdots + a_{n-1}\phi^{n-1},$$

then

$$g = u^{-1}\psi(a_0)\nu^0 + \cdots + u^{-1}\psi(a_{n-1})\nu^{n-1}$$

and therefore the classes also generate $K_\alpha(\Gamma)$. □

8 Invariant bases on flag manifolds

In this section, we use the same technique as above to produce a basis of invariant K-classes on the variety of complete flags in $\mathbb{C}^{n+1}$.

As in the previous section, let $T = (S^1)^{n+1}$ act componentwise on $\mathbb{C}^{n+1}$ by

$$(t_1, \dots, t_{n+1}) \cdot (z_1, \dots, z_{n+1}) = (t_1 z_1, \dots, t_{n+1} z_{n+1}),$$

and let $x_1, \dots, x_{n+1} \in \mathbb{Z}_T^*$ be the weights. This action induces a GKM action both on the flag manifold $Fl(\mathbb{C}^{n+1})$ and $\mathbb{C}P^n$.

The GKM graph (Γ, α) associated to $Fl(\mathbb{C}^{n+1})$ is the permutahedron: its vertices are in bijection with the elements of S_{n+1}, the group of permutations on $n+1$ elements, and there exists an edge e between two vertices σ and σ' if and only if σ and σ' differ by a transposition, i.e., $\sigma' = \sigma(i, j)$, for some $1 \leqslant i < j \leqslant n+1$; the axial function is given by $\alpha(\sigma, \sigma') = x_{\sigma(i)} - x_{\sigma'(i)}$.

The group S_{n+1} acts on the GKM graph by left multiplication on its vertices, and on $\mathfrak{t}^*$ by

$$\sigma \cdot x_i = x_{\sigma(i)}.$$

Using the identification (2.1), this action determines an action on $R(T)$ by defining

$$\sigma \cdot \left(e^{2\pi\sqrt{-1}x_i}\right) = e^{2\pi\sqrt{-1}x_{\sigma(i)}} \tag{8.1}$$

and then extending it to the elements of $R(T)$ in the natural way. Using the fiber bundle construction introduced in this paper we will now produce a basis of the K-ring of (Γ, α) composed of S_{n+1}-*invariant classes*, i.e., elements $f \in K_\alpha(\Gamma)$ satisfying

$$f(u) = u \cdot f(id) \quad \text{for every} \quad u \in S_{n+1}.$$

The main idea in the construction of our invariant basis of K-classes for $Fl(\mathbb{C}^{n+1})$ is to use the natural projection π of $Fl(\mathbb{C}^{n+1})$ to $\mathbb{C}P^n$ with fiber $Fl(\mathbb{C}^n)$ to construct these classes by an induction argument on n.

As we saw in Sect. 7, the GKM graph (Γ_B, α_B) associated to $\mathbb{C}P^n$ is $(\mathcal{K}_{n+1}, \alpha_B)$, where $\alpha_B(i, j) = x_i - x_j$ for every $1 \leq i \neq j \leq n+1$. The projection $\pi\colon Fl(\mathbb{C}^{n+1}) \to \mathbb{C}P^n$ can be described in a simple way in terms of the vertices of the GKM graphs; in fact $\pi\colon (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ is simply given by $\pi(\sigma) = \sigma(1)$, for every $\sigma \in S_{n+1}$.

Let z_i be $e^{2\pi\sqrt{-1}x_i}$ for every $i = 1, \dots, n+1$ and $\nu\colon [n+1] \to R(T)$ be invariant the K-class given by $\nu(i) = z_i$ for $i = 1, \dots, n+1$. By Proposition 7.4, the classes $1, \nu, \dots, \nu^n$ form an invariant basis of $K_{\alpha_B}(\mathcal{K}_{n+1})$, and they lift to S_{n+1}-invariant basic classes on Γ.

Let Γ_{n+1} be the fiber over $\mathbb{C} \cdot e_{n+1}$, where $\{e_1, \dots, e_{n+1}\}$ denotes the canonical basis of $\mathbb{C}^{n+1}$; the holonomy group on this fiber is isomorphic to S_n viewed as the subgroup of S_{n+1} which leaves the element $n+1$ fixed. Once we construct a basis of $K_\alpha(\Gamma_{n+1})$ consisting of holonomy invariant classes, we can extend those to a set of classes of $K_\alpha(\Gamma)$, which, together with the invariant basic classes constructed above, will generate a basis of $K_\alpha(\Gamma)$ as a module over $R(T)$. We will show that if we start with a natural choice for the base of the induction, then the global classes generated by this means are indeed S_{n+1} invariant.

Letting $I = [i_1, \dots, i_n]$ be a multi-index of non-negative integers, define

$$\mathbf{z}^I = z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$$

and let $C_I = C_T(\mathbf{z}^I)\colon S_{n+1} \to R(T)$ be the element defined by

$$C_I(\sigma) = \sigma \cdot \mathbf{z}^I \quad \text{for every} \quad \sigma \in S_{n+1};$$

it's easy to verify that C_I is an invariant class of $K_\alpha(\Gamma)$.

Theorem 8.1 *Let*

$$\mathcal{A}_n = \{I = [i_1, \ldots, i_n] \mid 0 \leqslant i_1 \leqslant n, 0 \leqslant i_2 \leqslant n-1, \ldots, 0 \leqslant i_n \leqslant 1\}.$$

Then the set

$$\{C_I = C_T(\mathbf{z}^I) \mid I \in \mathcal{A}_n\}$$

is an invariant basis of $K_\alpha(\Gamma)$ as an $R(T)$-module.

Proof As mentioned above, the proof is by induction. Let $n = 2$, then the fiber bundle $\pi : Fl(\mathbb{C}^3) \to \mathbb{C}P^2$ is a $\mathbb{C}P^1$-bundle. Let $p = \mathbb{C} \cdot e_3 \in \mathbb{C}P^2$ be the one dimensional subspace generated by the third vector in the canonical basis of $\mathbb{C}^3$. Then the fiber over p is a copy of $\mathbb{C}P^1$ and the invariant classes $C_{[0]}$ and $C_{[1]}$ form a basis of the K-ring of the fiber. We can extend these classes using transition maps between fibers, thus obtaining S_3-invariant K-classes $C_{[0,0]}$ and $C_{[0,1]}$ on $Fl(\mathbb{C}^3)$. By Proposition 7.4, the invariant classes 1, ν, ν^2 form a basis of the K-ring of the base, and they lift to S_3-invariant basic classes $C_{[1,0]}$ and $C_{[2,0]}$. By Theorem 5.3, the K-ring of $Fl(\mathbb{C}^3)$ is freely generated over $R(T)$ by the invariant classes $C_{[0,0]}, C_{[0,1]}, C_{[1,0]}, C_{[1,1]}, C_{[2,0]}$ and $C_{[2,1]}$.

The general statement follows by repeating inductively the same argument, since at each stage the fiber of $\pi : Fl(\mathbb{C}^{m+1}) \to \mathbb{C}P^m$ is a copy of $Fl(\mathbb{C}^m)$.

Remark 8.2 This argument can be adapted to give a basis of invariant K-classes for the generalized flag variety of type C_n. Let $\alpha_i = x_i - x_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = 2x_n$ be a choice of simple roots of type C_n. The corresponding Weyl group W is the group of signed permutations of n elements. Let $\Sigma = \{\alpha_2, \ldots, \alpha_n\}$, then $G/P(\Sigma)$ is a GKM manifold diffeomorphic to a complex projective space $\mathbb{C}P^{2n-1}$; its GKM graph is a complete graph $\mathcal{K}_{2n}$ whose vertices can be identified with the set $\{\pm 1, \ldots, \pm n\}$ and the axial function α is simply given by $\alpha(\pm i, \pm j) = \pm x_i \mp x_j$, for every edge $(\pm i, \pm j)$ of the GKM graph. Observe that the procedure in Sect. 7 can be used here to produce a W-invariant basis of $K_\alpha(\mathcal{K}_{2n})$. In fact it is sufficient to let $y_1 = x_1, \ldots, y_n = x_n$, $y_{n+1} = -x_1$ and $y_{2n} = -x_n$; the basis of Proposition 7.4 is S_{2n}-invariant, and hence in particular W-invariant.

Let G/B be the generic co-adjoint orbit of type C_n, and (Γ, α) its GKM graph. If we consider the natural projection $G/B \to G/P(\Sigma)$, the fiber is diffeomorphic to a generic co-adjoint orbit of type C_{n-1}. Hence, we can repeat the inductive argument used in type A_n to produce a W-invariant basis of $K_\alpha(\Gamma)$.

9 The Kostant–Kumar description

The manifolds in Sect. 4 are also describable in terms of compact groups. Namely, if we let G_0 be the compact form of G and K the maximal compact subgroup of P, then $M = G/P = G_0/K$. Moreover, $W = W_{G_0}$ and $W(\Sigma) = W_K$, W_{G_0}, and W_K being the Weyl groups of G_0 and K, so $M^T = W_{G_0}/W_K$. A fundamental theorem in equivariant K-theory is the Kostant–Kumar theorem, which asserts that $K_T(M)$ is isomorphic to the tensor product

$$R^{W_K} \otimes_{R^W} R, \tag{9.1}$$

where R is the character ring $R(T)$, and R^{W_K} and R^W are the subrings of W_K and W-invariant elements in R. This description of $K_T(M)$ generalizes to K-theory the well-known Borel description of the equivariant cohomology ring $H_T(M)$ as the tensor product

$$\mathbb{S}(\mathfrak{t}^*)^{W_K} \otimes_{\mathbb{S}(\mathfrak{t}^*)^W} \mathbb{S}(\mathfrak{t}^*) \tag{9.2}$$

and in [4] the authors showed how to reconcile this description with the GKM description of $H_T(M)$. Mutatis mutandi, their arguments work as well in K-theory and we will give below a brief description of the K-theoretic version of their theorem.

Let Γ be the GKM graph of M. As we pointed out above, $M^T = W/W_K$, so the vertices of Γ are the elements of W/W_K. Now let $f \otimes g$ be a decomposable element of the tensor product (9.1). Then one gets an R-valued function, $k(f \otimes g)$, on W/W_K by setting

$$k(f \otimes g)(wW_K) = wfg. \tag{9.3}$$

One can show that this defines a ring morphism, k, of the ring (9.1) into the ring $\mathrm{Maps}(M^T, R)$, and in fact that this ring morphism is a bijection of the ring (9.1) onto $K_\alpha(\Gamma)$. (For the proof of the analogous assertions in cohomology see Sect. 2.4 of [4].) Moreover the action of W on $K_T(M)$ becomes, under this isomorphism, the action

$$w(f_1 \otimes f_2) = f_1 \otimes wf_2 \tag{9.4}$$

of W on the ring (9.1), so the ring of W-invariant elements in $K_T(M)$ gets identified with the tensor product $R^{W_K} \otimes_{R^W} R^W$, which is just the ring R^{W_K} itself. Finally we note that if M is the generalized flag variety, $G/B = G_0/T$, (9.1) becomes the tensor product

$$R \otimes_{R^W} R \tag{9.5}$$

and the ring of W-invariant elements in $K_T(M)$ becomes R. Moreover, if π is the fibration $G_0/T \to G_0/K$, the fiber F over the identity coset of G_0/K is K/T; so $K_T(F)$ is the tensor product $R \otimes_{R^{W_K}} R$, and the subring of W_K-invariant elements in $K_T(F)$ is R which, as we saw above, is also the ring of W_G-invariant elements in $K_T(G_0/T)$ which is also (see Sect. 6) the ring of invariant elements associated with the fibration $G_0/T \to G_0/K$. Thus, most of the features of our GKM description of the fibration $G_0/T \to G_0/K$ have simple interpretations in terms of this Kostant–Kumar model.

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